THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 10 Problems
8th April 2024

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1. Consider the ring of Gaussian integer $(\mathbb{Z}[i], +, \times)$. Define the norm function $|\cdot| : \mathbb{Z}[i] \to \mathbb{Z}_{\geq 0}$ by $|a + bi| = a^2 + b^2$. Let $z, w \in \mathbb{Z}[i]$ so that $w \neq 0$, and $|z| \geq |w|$.
   (a) Show that there exists some $k, r \in \mathbb{Z}[i]$ so that $z = kw + r$, with $|r| < |w|$.
   (b) Hence, prove that $(\mathbb{Z}[i], +, \times)$ is a PID, i.e. every ideal $I \subset \mathbb{Z}[i]$ is generated by one single element. (Hint: Follow what we did for $\mathbb{Z}$ and $F[x]$.)
2. Is $\mathbb{Z}[x]$ a PID?
3. Prove that $\mathbb{Z}[i]/(1 + i) \cong \mathbb{Z}/2\mathbb{Z}$.
4. Prove that $\mathbb{Z}[i]/(2) \cong \mathbb{Z}_2[x]/(x^2)$.
5. (a) Determine how many elements are there in the quotient rings $\mathbb{Z}[i]/(3)$ and $\mathbb{Z}[i]/(5)$.
   (b) Show that $\mathbb{Z}[i]/(3)$ is a field but $\mathbb{Z}[i]/(5)$ is not.
   (c) Show that $\mathbb{Z}[i]/(p) \cong \mathbb{Z}[x]/(p, x^2 + 1) \cong \mathbb{Z}_p[x]/(x^2 + 1)$.
6. Determine whether the following polynomials are irreducible in $\mathbb{Z}_5[x]$:
   (i) $f(x) = x^3 + 2x + 1$, (ii) $g(x) = 2x^3 + x^2 + 2x + 2$.
7. Let $f(x) = x^4 + kx^2 + 1 \in \mathbb{Z}[x]$.
   (a) Suppose that $f(x)$ is reducible, prove that $f(x) = (x^2 + ax + b)(x^2 - ax + b)$, where $a \in \mathbb{Z}$ and $b = \pm 1$.
   (b) Show that if $f(x)$ is reducible, then $k = 2 - a^2$ or $k = -2 - a^2$. Hence, deduce that if $k > 1$, then $f(x)$ is irreducible.
   (c) Show that $f(x) = x^4 - 22x + 1$ is irreducible in $\mathbb{Z}[x]$.
   (d) Show that $f(x) = x^4 - 23x + 1$ is reducible in $\mathbb{Z}[x]$ by finding a factorization in $\mathbb{Z}[x]$.